LINEAR DETERMINANTAL EQUATIONS FOR ALL PROJECTIVE SCHEMES

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ABSTRACT. We prove that every projective embedding of a connected scheme determined by the complete linear series of a sufficiently ample line bundle is defined by the 2×2 minors of a 1-generic matrix of linear forms. Extending the work of Eisenbud-Koh-Stillman for integral curves, we also provide effective descriptions for such determinantally presented ample line bundles on products of projective spaces, Gorenstein toric varieties, and smooth varieties.

1. INTRODUCTION

Relating the geometric properties of a variety to the structural features of its defining equations is a fundamental challenge in algebraic geometry. Describing generators for the homogeneous ideal associated to a projective scheme is a basic form of this problem. For a rational normal curve, a Segre variety, or a quadratic Veronese variety, the homogeneous ideal is conveniently expressed as the 2-minors (i.e. the determinants of all 2×2 submatrices) of a generic Hankel matrix, a generic matrix, or a generic symmetric matrix respectively. These determinantal representations lead to a description of the minimal graded free resolution of the homogeneous ideal of the variety and equations for higher secant varieties. Mumford's "somewhat startling observation" in [M, p. 31] is that a suitable multiple of every projective embedding is the intersection of a quadratic Veronese variety with a linear space and, hence, defined by the 2-minors of a matrix of linear forms. Exercise 6.10 in [Ei2] rephrases this as a "(vague) principle that embeddings of varieties by sufficiently positive bundles are often defined by ideals of 2×2 minors". Our primary goal is to provide a precise form of this principle.

To be more explicit, consider a scheme X embedded in $\mathbb{P}^r$ by the complete linear series of a line bundle L. As in [EKS, p. 514], the line bundle L is called ***determinantally presented*** if the homogeneous ideal $I_{X|\mathbb{P}^r}$ of X in $\mathbb{P}^r$ is generated by the 2-minors of a 1-generic matrix (i.e. no conjugate matrix has a zero entry) of linear forms. Definition 3.1 in [Gr2] states that a property holds for a ***sufficiently ample*** line bundle on X if there exists a line bundle A such that the property holds for all $L \in \mathrm{Pic}(X)$ for which $L \otimes A^{-1}$ is ample. Our main result is:

Theorem 1.1. *Every sufficiently ample line bundle on a connected scheme is determinantally presented.*

We also describe, in terms of Castelnuovo-Mumford regularity, a collection of determinantally presented line bundles on an arbitrary projective scheme; see Corollary 3.3.

This theorem is a new incarnation of a well-known phenomenon — roughly speaking, the complexity of the first few syzygies of a projective subscheme is inversely related to the positivity of the corresponding linear series. Nevertheless, Theorem 1.1 counter-intuitively implies that most projective embeddings by a complete linear series are simply the intersection of a Segre variety with a linear subspace. More precisely, if we fix the Euclidean metric on the ample cone $\mathrm{Amp}(X)$ which it inherits from the finite-dimensional real vector space $N^1(X) \otimes \mathbb{R}$, then the fraction of determinantally presented ample classes within distance ρ of the trivial class approaches 1 has ρ tends to ∞.

Theorem 1.1 also has consequences beyond showing that the homogeneous ideal is generated by quadrics of rank at least 2. Proposition 6.13 in [Ei2] shows that an Eagon-Northcott complex is a direct summand of the minimal graded free resolution of the ideal. Despite the classic examples, being able to give a complete description of this resolution in the general setting seems overly optimistic. However, a determinantal presentation provides many equations for higher secant varieties; see Proposition 1.3 in [EKS]. For a scheme $X \subset \mathbb{P}^r$, let $\mathrm{Sec}^k(X)$ be the Zariski closure of the union of the linear spaces spanned by collections of $k+1$ points on X. A natural generalization of Theorem 1.1 would be:

Conjecture 1.2. *Let k be a positive integer. If $X \subset \mathbb{P}^r$ is embedded by the complete linear series of a sufficiently ample line bundle, then the homogeneous ideal of $\mathrm{Sec}^k(X)$ is generated by the $(k+2)$-minors of a 1-generic matrix of linear forms.*

This conjecture holds for rational normal curves (see Proposition 4.3 in [Ei1]), rational normal scrolls (see Proposition 2.2 in [C-J]), Segre varieties, and quadratic Veronese varieties (see [SS, §4]). It also extends the conjecture for curves appearing in [EKS, p. 518] for which [Rav] proves a set-theoretic version and [Gin, §7] proves a scheme-theoretic version. Although Theorem 1.1.4 in [BGL] produces counterexamples to this conjecture for some singular X, Corollary 1.2.4 in [BGL] provides supporting evidence when X is smooth. Theorem 1.3 in [BB] suggests that the secant varieties in Conjecture 1.2 should be replaced by cactus varieties.

The secondary goal of this article is to effectively bound the determinantally presented line bundles on specific schemes. For an integral curve of genus g, Theorem 1 in [EKS] shows that a line bundle is determinantally presented when its degree is at least $4g + 2$ and this bound is sharp. We provide the analogous result on smooth varieties and Gorenstein toric varieties:

Theorem 1.3. *Let X be a smooth variety of dimension n or an n-dimensional Gorenstein toric variety and let A be a very ample line bundle on X such that $(X,A) \neq \left(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}}^n(1)\right)$. If B is a nef line bundle, K_X is the dualizing bundle on X, and $L := K_X^2 \otimes A^j \otimes B$ with $j \geqslant 2n+2$, then L is determinantally presented.*

As an application of our methods, we describe determinantally presented ample line bundles on products of projective spaces; see Theorem 4.1.

To prove these theorems, we need a source of appropriate matrices. Composition of linear series (a.k.a. multiplication in total coordinate ring or the Cox ring) traditionally supply the required matrices. If $X \subset \mathbb{P}^r$ is embedded by the complete linear series for a line bundle

L, then $H^0(X, L)$ is the space of linear forms on $\mathbb{P}^r$. Factoring L as $L = E \otimes E'$ for some $E, E' \in \mathrm{Pic}(X)$ yields a natural map $\mu \colon H^0(X, E) \otimes H^0(X, E') \to H^0(X, E \otimes E') = H^0(X, L)$. By choosing ordered bases $y_1, \ldots, y_s \in H^0(X, E)$ and $z_1, \ldots, z_t \in H^0(X, E')$, we obtain an associated $(s \times t)$-matrix $\Omega := [\mu(y_i \otimes z_j)]$ of linear forms. The matrix Ω is 1-generic and its ideal $\mathrm{I}_2(\Omega)$ of 2-minors vanishes on X; see Proposition 6.10 in [Ei2]. Numerous classic examples of this construction can be found in [Roo].

With these preliminaries, the problem reduces to finding conditions on E and E' which guarantee that $I_{X|\mathbb{P}^r} = \mathrm{I}_2(\Omega)$. Inspired by the approach in [EKS], Theorem 3.2 achieves this by placing restrictions on certain modules arising from the line bundles L, E, and E'. The key hypotheses require these modules to have a ***linear free presentation***; the generators of the $\mathbb{N}$-graded modules have degree 0 and their first syzygies must have degree 1. Methods introduced by Green and Lazarsfeld [Gr1, GL1] (for an expository account see [Ei2, §8], [Gr3], or [La1, §1]) yield a cohomological criterion for our modules to have a linear free presentation. Hence, we can prove Theorem 1.1 by combining this with uniform vanishing results derived from Castelnuovo-Mumford regularity. Building on known conditions (i.e. sufficient conditions for a line bundle to satisfy N_1), we obtain effective criteria for the appropriate modules to have a linear free presentation on Gorenstein toric varieties, and smooth varieties.

Rather than focusing exclusively on a single factorization of the line bundle L, we set up the apparatus to handle multiple factorizations; see Lemma 3.1. Multiple factorizations of a line bundle were used in [BH] to study the equations and syzygies of elliptic normal curves and their secant varieties. They also provide a geometric interpretation for the *flattenings* appearing in [GSS, §7] and [CGG, p. 1915]. Using this more general setup, we are able to describe the homogeneous ideal for every embedding of a product of projective spaces by a very ample line bundle as the 2-minors of appropriate 1-generic matrices of linear forms; see Proposition 4.4.

CONVENTIONS. In this paper, $\mathbb{N}$ is the set of nonnegative integers, $\mathbb{1}_W \in \mathrm{Hom}(W, W)$ is the identity map, and $\mathbf{1} := (1, \ldots, 1)$ is the vector in which every entry is 1. We work over an algebraically closed field $\Bbbk$ of characteristic zero. A variety is always irreducible and all of our toric varieties are normal. For a vector bundle U, we write U^j for the j-fold tensor product $U^{\otimes j} = U \otimes \cdots \otimes U$.

2. LINEAR FREE PRESENTATIONS

This section collects the criteria needed to show that certain modules arising from line bundles have a linear free presentation. While accomplishing this, we also establish some notation and nomenclature used throughout the document.

Let X be a projective scheme over $\Bbbk$, let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module, and let L be a line bundle on X. We write $\Gamma(L) := H^0(X,L)$ for the $\Bbbk$-vector space of global sections and $S := \mathrm{Sym}\big(\Gamma(L)\big)$ for the homogeneous coordinate ring of $\mathbb{P}^r := \mathbb{P}\big(\Gamma(L)\big)$. Consider the $\mathbb{N}$-graded S-module $F := \bigoplus_{j \geqslant 0} H^0(X, \mathscr{F} \otimes L^j)$. When $\mathscr{F} = \mathscr{O}_X$, F is the section ring of L. However, when $\mathscr{F} = L$, the module F is the truncation of the section ring omitting the zeroth graded piece and shifting degrees by -1. Let $P_\bullet$ be a minimal graded free resolution of F:

$$\cdots \longrightarrow \underset{P_i}{\underbrace{\bigoplus S(-a_{i,j})}} \longrightarrow \cdots \longrightarrow \underset{P_1}{\underbrace{\bigoplus S(-a_{1,j})}} \longrightarrow \underset{P_0}{\underbrace{\bigoplus S(-a_{0,j})}} \longrightarrow F \longrightarrow 0.$$

Following [EKS, p. 515], we say that, for $p \in \mathbb{N}$, $\mathscr{F}$ has a **linear free resolution to stage** p **with respect to** L or F has a **linear free resolution to stage** p if $P_i = \bigoplus S(-i)$ for all $0 \leqslant i \leqslant p$. Thus, F has a linear free resolution to stage 0 if and only if it is generated in degree 0. Since having a linear free resolution to stage 1 implies that the relations among the generators (a.k.a. first syzygies) are linear, the module F has a linear free resolution to stage 1 if and only if it has a linear free presentation. In this case, we say that $\mathscr{F}$ has a **linear free presentation with respect to** L. More generally, having a linear free resolution to stage p is the module-theoretic analogue of the N_p-property introduced in [GL1, §3]. If X is connected, then the line bundle L satisfies N_1 precisely when L has a linear free presentation with respect to itself and satisfies N_p when L has a linear free resolution to stage p. Following Convention 0.4 in [EL], we do not assume that X is normal.

Henceforth, we assume that L is globally generated. In other words, the natural evaluation map $\mathrm{ev}_L \colon \Gamma(L) \otimes_\Bbbk \mathscr{O}_X \longrightarrow L$ is surjective. If $M_L := \mathrm{Ker}(\mathrm{ev}_L)$, then M_L is a vector bundle of rank $r := \dim_\Bbbk \Gamma(L) - 1$ which sits in the short exact sequence

$$0 \longrightarrow M_L \longrightarrow \Gamma(L) \otimes_\Bbbk \mathscr{O}_X \longrightarrow L \longrightarrow 0. \qquad (*)$$

For convenience, we record the following cohomological criteria which is a minor variant of Theorem 5.6 in [Ei2], Proposition 2.4 in [Gr3], or Lemma 1.6 in [EL].

Lemma 2.1. *If $H^1(X, \bigwedge^i M_L \otimes \mathscr{F} \otimes L^j) = 0$ for all $1 \leqslant i \leqslant p+1$ and all $j \geqslant 0$, then the coherent $\mathscr{O}_X$-module $\mathscr{F}$ has a linear free resolution to stage p with respect to L. In characteristic zero, $\bigwedge^i M_L$ is a direct summand of M_L^i, so it suffices to show $H^1(X, M_L^i \otimes \mathscr{F} \otimes L^j) = 0$ for all $1 \leqslant i \leqslant p+1$ and all $j \geqslant 0$.* $\qquad\square$

Sketch of Proof. The key observation is that the graded Betti numbers for the minimal free resolution of F can be computed via Koszul cohomology. If L is globally generated and $\mathbb{P}^r = \mathbb{P}\big(H^0(X,L)\big)$, then there is a morphism $\varphi_L \colon X \longrightarrow \mathbb{P}^r$ with $\varphi_L^*\big(\mathscr{O}_{\mathbb{P}^r}(1)\big) = L$. Since the pullback by φ_L^* of $0 \longrightarrow M_{\mathscr{O}_{\mathbb{P}^r}(1)} \longrightarrow \Gamma\big(\mathscr{O}_{\mathbb{P}^r}(1)\big) \otimes_\Bbbk \mathscr{O}_{\mathbb{P}^r} \longrightarrow \mathscr{O}_{\mathbb{P}^r}(1) \longrightarrow 0$ is just $(*)$, the proof of Theorem 5.6 in [Ei2] goes through working on X instead of $\mathbb{P}^r$. $\qquad\square$

Multigraded Castelnuovo-Mumford regularity, as developed in [MS, §6] or [HSS, §2], allows us to exploit this criteria. To be more precise, fix a list $B_1, \ldots, B_\ell$ of globally generated line bundles on X. For a vector $\boldsymbol{u} := (u_1, \ldots, u_\ell) \in \mathbb{Z}^\ell$, we set $\boldsymbol{B^u} := B_1^{u_1} \otimes \cdots \otimes B_\ell^{u_\ell}$ and we write $\mathfrak{B} := \{\boldsymbol{B^u} : \boldsymbol{u} \in \mathbb{N}^\ell\} \subset \mathrm{Pic}(X)$ for the submonoid generated by these line bundles. If

$e_1, \ldots, e_\ell$ is the standard basis for $\mathbb{Z}^\ell$ then $\boldsymbol{B}^{\boldsymbol{e}_j} = B_j$. A coherent $\mathscr{O}_X$-module $\mathscr{F}$ is said to be ***regular with respect to*** $B_1, \ldots, B_\ell$ if $H^i(X, \mathscr{F} \otimes \boldsymbol{B}^{-\boldsymbol{u}}) = 0$ for all $i > 0$ and all $\boldsymbol{u} \in \mathbb{N}^\ell$ satisfying $|\boldsymbol{u}| := u_1 + \cdots + u_\ell = i$. When $\ell = 1$, we recover the version of Castelnuovo-Mumford regularity found in [La2, §1.8].

Although the definition may not be intuitive, the next result shows that regular line bundles are at least ubiquitous.

Lemma 2.2. *Let X be a scheme and let $B_1, \ldots, B_\ell$ be globally generated line bundles on X. If there is a positive vector $\boldsymbol{w} \in \mathbb{Z}^\ell$ such that $\boldsymbol{B}^{\boldsymbol{w}}$ is ample, then a sufficiently ample line bundle on X is regular with respect to $B_1, \ldots, B_\ell$.*

The hypothesis on $\boldsymbol{w}$ means that the cone $\mathrm{pos}(B_1, \ldots, B_\ell)$ generated by $B_1, \ldots, B_\ell$ contains an ample line bundle. In other words, the subcone $\mathrm{pos}(B_1, \ldots, B_\ell)$ of $\mathrm{Nef}(X)$ has a nonempty intersection with the interior of $\mathrm{Nef}(X)$.

Proof. It suffices to find a line bundle A on X such that, for any nef line bundle C, $A \otimes C$ is regular with respect to $B_1, \ldots, B_\ell$. Because $\boldsymbol{B}^{\boldsymbol{w}}$ is ample, Fujita's Vanishing Theorem (e.g. Theorem 1 in [Fu2]) implies that there is $k \in \mathbb{N}$ such that, for any nef line bundle C, we have $H^i(X, \boldsymbol{B}^{j\boldsymbol{w}} \otimes C) = 0$ for all $i > 0$ and all $j \geqslant k$. Let $n := \dim X$ and consider $A := \boldsymbol{B}^{(k+n)\boldsymbol{w}}$. Since $\boldsymbol{w}$ is positive, the line bundle $\boldsymbol{B}^{n\boldsymbol{w}-\boldsymbol{u}}$ is nef for all $\boldsymbol{u} \in \mathbb{N}^\ell$ with $0 \leqslant |\boldsymbol{u}| \leqslant n$. Therefore, we have $H^i(X, (A \otimes C) \otimes \boldsymbol{B}^{-\boldsymbol{u}}) = H^i(X, \boldsymbol{B}^{k\boldsymbol{w}} \otimes (\boldsymbol{B}^{n\boldsymbol{w}-\boldsymbol{u}} \otimes C)) = 0$ for all $i > 0$ and all $\boldsymbol{u} \in \mathbb{N}^\ell$ satisfying $|\boldsymbol{u}| = i$. $\qquad \square$

Before describing the pivotal results in this section, we record a technical lemma bounding the regularity of certain tensor products. Our approach is a hybrid of Proposition 1.8.9 and Remark 1.8.16 in [La2].

Lemma 2.3. *Let X be a scheme of dimension n and $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. Fix a vector bundle V and a globally generated ample line bundle B on X. If m is positive integer such that $\mathscr{F}$, V, and B^m are all regular with respect to B, then $\mathscr{F} \otimes V \otimes B^w$ is also regular with respect to B for all $w \geqslant (m-1)(n-1)$.*

Proof. Since $\mathscr{F}$ and B^m are regular with respect to B, Corollary 3.2 in [Ara] or Theorem 7.8 in [MS] (cf. Proposition 1.8.8 in [La2]) produces a locally free resolution of $\mathscr{F}$ of the form $\cdots \longrightarrow \bigoplus B^{-jm} \longrightarrow \cdots \longrightarrow \bigoplus B^{-m} \longrightarrow \bigoplus \mathscr{O}_X \longrightarrow \mathscr{F} \longrightarrow 0$. Tensoring by a locally free sheaf perserves exactness, so we obtain the exact complex

$$\cdots \longrightarrow \bigoplus V \otimes B^{w-jm} \longrightarrow \cdots \longrightarrow \bigoplus V \otimes B^w \longrightarrow \mathscr{F} \otimes V \otimes B^w \longrightarrow 0.$$

Since V is also regular with respect to B, Mumford's Lemma (e.g. Theorem 1.8.5 in [La2]) implies that $H^{i+j}(X, V \otimes B^{w-jm-i}) = 0$ for $i \geqslant 1$ provided we have $w - jm - i \geqslant -i - j$. Chasing through the complex (see Proposition B.1.2 in [La2]), we conclude that $\mathscr{F} \otimes V \otimes B^w$ is also regular with respect to B when $w \geqslant (m-1)(n-1)$. $\qquad \square$

The following three propositions each provide sufficient conditions for an appropriate line bundle to have a linear free presentation with respect to another line bundle.

Proposition 2.4. *Fix a positive integer m and a scheme X of dimension n. Let L be a line bundle on X and let B be a globally generated ample line bundle on X. If L^j and B^m are regular with respect to B for all $j \geqslant 1$, then B^w has a linear free presentation with respect to L for all $w \geqslant 2(m-1)n+1$.*

Proof. We first prove that $M_L \otimes B^m$ is regular with respect to B. Tensoring $(*)$ with B^{m-i} and taking the associated long exact sequence gives

$$\Gamma(L) \otimes H^0(X, B^{m-i}) \longrightarrow H^0(X, L \otimes B^{m-i}) \longrightarrow H^1(X, M_L \otimes B^{m-i}) \longrightarrow \cdots$$
$$\longrightarrow H^{i-1}(X, L \otimes B^{m-i}) \longrightarrow H^i(X, M_L \otimes B^{m-i}) \longrightarrow \Gamma(L) \otimes H^i(X, B^{m-i}).$$

Since L is regular with respect to B, Mumford's Lemma (e.g. Theorem 1.8.5 in [La2]) shows that, for all $k \in \mathbb{N}$, the map $\Gamma(L) \otimes H^0(X, B^k) \longrightarrow H^0(X, L \otimes B^k)$ is surjective and, for all $i > 0$ and all $k \in \mathbb{N}$, we have $H^i(X, L \otimes B^{k-i}) = 0$. As m is a positive integer, the map $\Gamma(L) \otimes H^0(X, B^{m-1}) \longrightarrow H^0(X, L \otimes B^{m-1})$ is surjective and $H^{i-1}(X, L \otimes B^{m-i}) = 0$ for all $i > 1$. Since B^m is also regular with respect to B, we have $H^i(X, B^{m-i}) = 0$ for all $i > 0$. It follows that $H^i(X, M_L \otimes B^{m-i}) = 0$ for all $i > 0$.

By Lemma 2.1, it suffices to show that, for all $j \in \mathbb{N}$, we have $H^1(X, M_L \otimes B^w \otimes L^j) = 0$ and $H^1(X, M_L^2 \otimes B^w \otimes L^j) = 0$. Thus, it suffices to show that the vector bundles $M_L \otimes B^{w+1} \otimes L^j$ and $M_L^2 \otimes B^{w+1} \otimes L^j$ are both regular with respect to B. If $w \geqslant (m-1)n$, then Lemma 2.3 implies that $(M_L \otimes B^m) \otimes L^j \otimes B^{w+1-m} = M_L \otimes B^{w+1} \otimes L^j$ is regular with respect to B. Similarly, if $w \geqslant 2(m-1)n+1$, then using Lemma 2.3 twice establishes that the vector bundle

$$\left((M_L \otimes B^m) \otimes (M_L \otimes B^m) \otimes B^{(m-1)(n-1)} \right) \otimes L^j \otimes B^{w-mn-m+n} = M_L^2 \otimes B^{w+1} \otimes L^j$$

is also regular with respect to B. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

By adapting the proof of Theorem 1.1 in [HSS], we obtain the second proposition.

Proposition 2.5. *Let $m \in \mathbb{N}^\ell$ be a vector satisfying $B^{m-e_j} \in \mathfrak{B}$ for all $1 \leqslant j \leqslant \ell$ and let the coherent $\mathscr{O}_X$-module $\mathscr{F}$ be regular with respect to $B_1, \ldots, B_\ell$. If $L := B^m$ and the map*

$$\Gamma(L) \otimes H^0(X, \mathscr{F} \otimes B^{-e_j}) \longrightarrow H^0(X, \mathscr{F} \otimes B^{m-e_j})$$

is surjective for all $1 \leqslant j \leqslant \ell$, then $\mathscr{F}$ has a linear presentation with respect to L.

The condition that $B^{m-e_j} \in \mathfrak{B}$ for all $1 \leqslant j \leqslant \ell$ implies that $L = B^m$ lies in the interior of the cone $\mathrm{pos}(B_1, \ldots, B_\ell)$.

Proof. We first prove that $M_L \otimes \mathscr{F}$ is regular with respect to $B_1, \ldots, B_\ell$. Tensoring $(*)$ with $\mathscr{F} \otimes B^{-u}$ and taking the associated long exact sequence gives

$$\Gamma(L) \otimes H^0(X, \mathscr{F} \otimes B^{-u}) \longrightarrow H^0(X, \mathscr{F} \otimes B^{m-u}) \longrightarrow H^1(X, M_L \otimes \mathscr{F} \otimes B^{-u}) \longrightarrow \cdots$$
$$\longrightarrow H^{i-1}(X, \mathscr{F} \otimes B^{m-u}) \longrightarrow H^i(X, M_L \otimes \mathscr{F} \otimes B^{-u}) \longrightarrow \Gamma(L) \otimes H^i(X, \mathscr{F} \otimes B^{-u}).$$

Since $\mathscr{F}$ is regular with respect to $B_1, \ldots, B_\ell$, Theorem 2.1 in [HSS] shows that, for all $i > 0$ and all $u, v \in \mathbb{N}^\ell$ with $|u| = i$, $H^i(X, \mathscr{F} \otimes B^{v-u}) = 0$. As $B^{m-e_j} \in \mathfrak{B}$ for $1 \leqslant j \leqslant \ell$, we see that $H^{i-1}(X, \mathscr{F} \otimes B^{m-u}) = 0$ for all $i > 1$ and all $u \in \mathbb{N}^\ell$ satisfying $|u| = i$. By hypothesis,

the map $\Gamma(L) \otimes H^0(X, \mathscr{F} \otimes B^{-e_j}) \longrightarrow H^0(X, \mathscr{F} \otimes B^{m-e_j})$ is surjective for all $1 \leqslant j \leqslant \ell$. It follows that $H^i(X, M_L \otimes \mathscr{F} \otimes B^{-u}) = 0$ for all $i > 0$ and all $u \in \mathbb{N}^\ell$ such that $|u| = i$.

By Lemma 2.1, it suffices to show that $H^1(X, M_L \otimes \mathscr{F} \otimes L^j)$ and $H^1(X, M_L^2 \otimes \mathscr{F} \otimes L^j)$ are zero for $j \in \mathbb{N}$. Since $M_L \otimes \mathscr{F}$ is regular with respect to $B_1, \ldots, B_\ell$, the vanishing of the first group follows from Theorem 2.1 in [HSS]. For the second, tensoring $(*)$ with $M_L \otimes \mathscr{F} \otimes L^j$ gives the exact sequence:

$$\Gamma(L) \otimes H^0(X, M_L \otimes \mathscr{F} \otimes L^j) \longrightarrow H^0(X, M_L \otimes \mathscr{F} \otimes L^{j+1}) \longrightarrow H^1(X, M_L^2 \otimes \mathscr{F} \otimes L^j) \longrightarrow 0.$$

Because $M_L \otimes \mathscr{F}$ is regular with respect to $B_1, \ldots, B_\ell$, Theorem 2.1 in [HSS] also shows that the left map is surjective for all $j \geqslant 0$. $\square$

Our third proposition is a variant of Proposition 3.1 in [EL].

Proposition 2.6. *Let X be a smooth variety of dimension n, let K_X be its canonical bundle, and let A be a very ample line bundle on X such that $(X, A) \neq (\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1))$. Suppose that B and C are nef line bundles on X. If the integers w and m are both greater than n, then the line bundle $K_X \otimes A^w \otimes B$ has a linear free presentation with respect to $K_X \otimes A^m \otimes C$.*

Proof. Let $\mathscr{F} := K_X \otimes A^w \otimes B$ and $L := K_X \otimes A^m \otimes C$. Since Proposition 3.1 in [EL] shows that L satisfies N_0 and Equation 3.2 in [EL] shows that $H^1(X, M_L^i \otimes \mathscr{F} \otimes L^j) = 0$ for all $1 \leqslant i \leqslant 2$ and all $j \geqslant 0$, Lemma 2.1 completes the proof. $\square$

3. DETERMINANTALLY PRESENTED LINE BUNDLES

The goal of this section is to prove Theorem 1.1. We realize this goal by developing general methods for showing that a line bundle is determinantally presented; see Theorem 3.2.
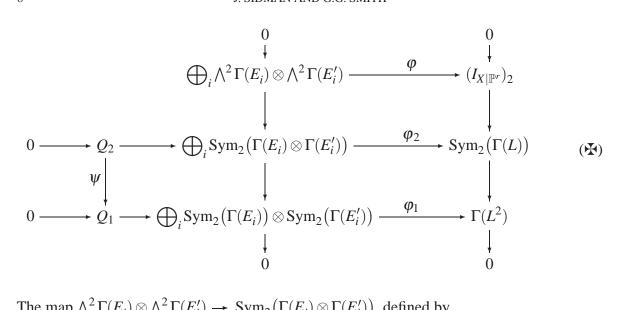
Suppose that $X \subset \mathbb{P}^r$ is embedded by the complete linear series for a line bundle L. Factor L as $L = E \otimes E'$ for some $E, E' \in \mathrm{Pic}(X)$ and let $\mu_{E, E'} \colon H^0(X, E) \otimes H^0(X, E') \longrightarrow H^0(X, L)$ denote the natural multiplication map. Choose ordered bases $y_1, \ldots, y_s$ and $z_1, \ldots, z_t$ for the $\Bbbk$-vector spaces $H^0(X, E)$ and $H^0(X, E')$ respectively. Define $\Omega = \Omega(E, E')$ to be the associated $(s \times t)$-matrix $[\mu_{E, E'}(y_i \otimes z_j)]$ of linear forms. Its ideal $\mathrm{I}_2(\Omega)$ of 2-minors is independent of the choice of bases. Proposition 6.10 in [Ei2] shows that Ω is 1-generic and that $\mathrm{I}_2(\Omega)$ vanishes on X.

Inspired by [EKS, §2], the key technical result is:

Lemma 3.1. *If L is a very ample line bundle on X satisfying N_1 and $\{(E_i, E_i')\}$ is a family of factorizations for L, then the commutative diagram $(\maltese)$ has exact rows and columns. Moreover, if φ_2 is surjective, then the homogeneous ideal $I_{X|\mathbb{P}^r}$ is generated by the 2-minors of the matrices $\Omega(E_i, E_i')$ if and only if Q_2 surjects onto Q_1.*

Proof. To begin, we prove the columns are exact. Since L satisfies N_0 (i.e. the natural maps $\mathrm{Sym}_j(\Gamma(L)) \longrightarrow H^0(X, L^j)$ are surjective for all $j \in \mathbb{N}$), the ideal $I_{X|\mathbb{P}^r}$ is the kernel of the map from the homogeneous coordinate ring of $\mathbb{P}^r$ to the section ring of L. By taking the quadratic components, we obtain the right column. The middle column is the direct sum of the complexes:

$$0 \longrightarrow \wedge^2 \Gamma(E_i) \otimes \wedge^2 \Gamma(E_i') \longrightarrow \mathrm{Sym}_2(\Gamma(E_i) \otimes \Gamma(E_i')) \longrightarrow \mathrm{Sym}_2(\Gamma(E_i)) \otimes \mathrm{Sym}_2(\Gamma(E_i')) \longrightarrow 0.$$

$$
\begin{array}{ccc}
 & 0 & 0 \\
 & \downarrow & \downarrow \\
\bigoplus_i \wedge^2 \Gamma(E_i) \otimes \wedge^2 \Gamma(E_i') & \xrightarrow{\;\varphi\;} & (I_{X|\mathbb{P}^r})_2 \\
\downarrow & & \downarrow \\
0 \to Q_2 \to \bigoplus_i \mathrm{Sym}_2\big(\Gamma(E_i) \otimes \Gamma(E_i')\big) & \xrightarrow{\;\varphi_2\;} & \mathrm{Sym}_2\big(\Gamma(L)\big) \\
{\scriptstyle\psi}\downarrow \quad\quad \downarrow & & \downarrow \\
0 \to Q_1 \to \bigoplus_i \mathrm{Sym}_2\big(\Gamma(E_i)\big) \otimes \mathrm{Sym}_2\big(\Gamma(E_i')\big) & \xrightarrow{\;\varphi_1\;} & \Gamma(L^2) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\qquad (\maltese)
$$

The map $\wedge^2 \Gamma(E_i) \otimes \wedge^2 \Gamma(E_i') \to \mathrm{Sym}_2\big(\Gamma(E_i) \otimes \Gamma(E_i')\big)$, defined by

$$
e \wedge f \otimes e' \wedge f' \longmapsto (e \otimes e') \cdot (f \otimes f') - (e \otimes f') \cdot (f \otimes e'), \tag{$\dagger$}
$$

is simply the inclusion map determined by the 2-minors of the generic matrix. The map $\mathrm{Sym}_2\big(\Gamma(E_i) \otimes \Gamma(E_i')\big) \to \mathrm{Sym}_2\big(\Gamma(E_i)\big) \otimes \mathrm{Sym}_2\big(\Gamma(E_i')\big)$ is $(e \otimes e') \cdot (f \otimes f') \mapsto ef \otimes e'f'$. Hence, each of these complexes is exact, so the middle column also is. By definition, Q_1 and Q_2 are the kernels of the φ_1 and φ_2 respectively, and ψ is the induced map between them.

We next identify the horizontal maps. By applying the functor Sym_2 to $\mu_{E_i,E_i'}$, we obtain a map from $\mathrm{Sym}_2\big(\Gamma(E_i) \otimes \Gamma(E_i')\big)$ to $\mathrm{Sym}_2\big(\Gamma(L)\big)$ for each i; φ_2 is their direct sum. The composite map $\mu_{L,L} \circ (\mu_{E_i,E_i'} \otimes \mu_{E_i,E_i'}) \colon \Gamma(E_i) \otimes \Gamma(E_i') \otimes \Gamma(E_i) \otimes \Gamma(E_i') \to \Gamma(L^2)$ factors through $\mathrm{Sym}_2\big(\Gamma(E_i)\big) \otimes \mathrm{Sym}_2\big(\Gamma(E_i')\big)$, and φ_1 is the direct sum of the associated maps from $\mathrm{Sym}_2\big(\Gamma(E_i)\big) \otimes \mathrm{Sym}_2\big(\Gamma(E_i')\big)$ to $\Gamma(L^2)$. The map φ is induced by φ_2. From ($\dagger$), we see that the image of φ is generated by the 2-minors of the matrices $\Omega(E_i, E_i')$.

Finally, the line bundle L satisfies N_1, so the quadratic component $(I_{X|\mathbb{P}^r})_2$ generates the entire ideal $I_{X|\mathbb{P}^r}$. Hence, the image of φ generates the ideal $I_{X|\mathbb{P}^r}$ if and only if φ is surjective. Since φ_2 is surjective, the Snake Lemma (e.g. Lemma 1.3.2 in [Wei]) shows that the surjectivity of φ is equivalent to the surjectivity of ψ. $\qquad\square$

Our main application for Lemma 3.1 focuses on a single factorization of the line bundle L. The proof follows the strategy in [EKS, §2].

Theorem 3.2. *Let L be a very ample line bundle on a scheme X satisfying N_1. If $L = E \otimes E'$ for some nontrivial $E, E' \in \mathrm{Pic}(X)$ and the following conditions hold*

  (a) *E has a linear presentation with respect to E',*
  (b) *E' has a linear presentation with respect to E,*
  (c) *E^2 has a linear presentation with respect to E',*
  (d) *both E and E' satisfy N_0,*

then the 2-minors of the matrix $\Omega(E, E')$ generate the homogeneous ideal of X in $\mathbb{P}\big(\Gamma(L)\big)$. In particular, the line bundle L is determinantally presented.

Proof. Given Lemma 3.1, it suffices to show that the map $\psi\colon Q_2 \longrightarrow Q_1$ is surjective. To accomplish this, we reinterpret both modules. Since Condition (a) or (b) imply that the map $\mu_{E,E'}\colon \Gamma(E) \otimes \Gamma(E') \to \Gamma(L)$ is surjective, we obtain an exact sequence

$$\mathrm{Ker}(\mu_{E,E'}) \otimes \Gamma(E) \otimes \Gamma(E') \longrightarrow \mathrm{Sym}_2\big(\Gamma(E) \otimes \Gamma(E')\big) \longrightarrow \mathrm{Sym}_2\big(\Gamma(L)\big) \longrightarrow 0\,,$$

so the image of $\mathrm{Ker}(\mu_{E,E'}) \otimes \Gamma(E) \otimes \Gamma(E')$ generates Q_2 in $\mathrm{Sym}_2\big(\Gamma(E) \otimes \Gamma(E')\big)$. The maps $\mu_{E,E}$ and $\mu_{E',E'}$ factor through $\mathrm{Sym}_2\big(\Gamma(E)\big)$ and $\mathrm{Sym}_2\big(\Gamma(E')\big)$ and thus induce maps $\eta\colon \mathrm{Sym}_2\big(\Gamma(E)\big) \to \Gamma(E^2)$ and $\eta'\colon \mathrm{Sym}_2\big(\Gamma(E')\big) \to \Gamma(E'^2)$ respectively. It follows that φ_1 is the composition $\mu_{E^2,E'^2} \circ (\eta \otimes \eta')\colon \mathrm{Sym}_2\big(\Gamma(E)\big) \otimes \mathrm{Sym}_2\big(\Gamma(E')\big) \to \Gamma(E^2 \otimes E'^2) = \Gamma(L^2)$. Hence, Q_1 is the sum of the images of $\mathrm{Ker}(\eta) \otimes \Gamma(E') \otimes \Gamma(E')$ and $\Gamma(E) \otimes \Gamma(E) \otimes \mathrm{Ker}(\eta')$, and the pullback to $\mathrm{Sym}_2\big(\Gamma(E)\big) \otimes \mathrm{Sym}_2\big(\Gamma(E')\big)$ of $\mathrm{Ker}(\mu_{E^2,E'^2})$.

We now break the proof that Q_2 surjects onto Q_1 into four steps:

  (i) the image of $\mathrm{Ker}(\mu_{E^2,E'}) \otimes \Gamma(E')$ in $\Gamma(E^2) \otimes \Gamma(E'^2)$ contains $\mathrm{Ker}(\mu_{E^2,E'^2})$,

  (ii) the image of $\mathrm{Ker}(\mu_{E,E'}) \otimes \Gamma(E)$ in $\Gamma(E^2) \otimes \Gamma(E')$ contains $\mathrm{Ker}(\mu_{E^2,E'})$,

  (iii) the image of $\mathrm{Ker}(\mu_{E,E'}) \otimes \Gamma(E)$ in $\mathrm{Sym}_2\big(\Gamma(E)\big) \otimes \Gamma(E')$ contains $\mathrm{Ker}(\eta) \otimes \Gamma(E')$,

  (iv) the image of $\mathrm{Ker}(\mu_{E,E'}) \otimes \Gamma(E')$ in $\Gamma(E) \otimes \mathrm{Sym}_2\big(\Gamma(E')\big)$ contains $\Gamma(E) \otimes \mathrm{Ker}(\eta')$.

Tensoring with $\Bbbk$-vector space $\Gamma(E')$, Step (ii) yields a surjective map

$$\mathrm{Ker}(\mu_{E,E'}) \otimes \Gamma(E) \otimes \Gamma(E') \longrightarrow \mathrm{Ker}(\mu_{E^2,E'}) \otimes \Gamma(E')\,.$$

Combining this with Step (i) shows that the map $\mathrm{Ker}(\mu_{E,E'}) \otimes \Gamma(E) \otimes \Gamma(E') \to \mathrm{Ker}(\mu_{E^2,E'^2})$ is surjective. Again by tensoring with $\Bbbk$-vector space $\Gamma(E')$, Step (iii) gives a surjective map $\mathrm{Ker}(\mu_{E,E'}) \otimes \Gamma(E) \otimes \Gamma(E') \to \mathrm{Ker}(\eta) \otimes \Gamma(E') \otimes \Gamma(E')$. Similarly, Step (iv) implies that the map $\mathrm{Ker}(\mu_{E,E'}) \otimes \Gamma(E) \otimes \Gamma(E') \longrightarrow \Gamma(E) \otimes \Gamma(E) \otimes \mathrm{Ker}(\eta')$ is surjective. Therefore, it is enough to establish the four steps.

For Step (i), Condition (c) implies that $\mathrm{Ker}(\mu_{E^2,E'})$, the span of the linear relations on $\bigoplus_{j \geqslant 0} H^0(X, E^2 \otimes E'^j)$ regarded as a $\mathrm{Sym}\big(\Gamma(E')\big)$-module, generates the relations in higher degrees as well. Hence, $\mathrm{Ker}(\mu_{E^2,E'}) \otimes \Gamma(E')$ maps onto the quadratic relations which are the kernel of the composite map $\mu_{E^2,E'^2} \circ (\mathbb{1}_{\Gamma(E^2)} \otimes \eta')$. Since this kernel is generated by $\Gamma(E^2) \otimes \mathrm{Ker}(\eta')$ and the pullback of $\mathrm{Ker}(\mu_{E^2,E'^2})$, Condition (d) implies that η' is surjective, and we have established Step (i).

To complete the proof, we simultaneously establish Steps (ii) and (iii); the symmetric argument yields Step (iv). Condition (b) implies that $\mathrm{Ker}(\mu_{E,E'})$ generates all the relations on $\bigoplus_{j \geqslant 0} H^0(X, E' \otimes E^j)$ regarded as a $\mathrm{Sym}\big(\Gamma(E)\big)$-module. In particular, the vector space $\mathrm{Ker}(\mu_{E,E'}) \otimes \Gamma(E)$ maps onto the quadratic relations which are the kernel of the composite map $\mu_{E^2,E'} \circ (\eta \otimes \mathbb{1}_{\Gamma(E')})$. This kernel is generated by $\mathrm{Ker}(\eta) \otimes \Gamma(E')$ and the pullback of $\mathrm{Ker}(\mu_{E^2,E'})$. Condition (d) implies that η is surjective, so Step (ii) and Step (iii) follow. $\square$

As the proof indicates, Theorem 3.2 holds under a weaker version of Condition (d). Specifically, it is only necessary that η and η' are surjective. Nevertheless, in all of our applications, a stronger condition is satisfied: both E and E satisfy N_1.

This theorem leads to a description, given in terms of Castelnuovo-Mumford regularity, for some determinantally presented line bundles on any projective scheme.

Corollary 3.3. *Let X be a connected scheme and let $B_1,\ldots,B_\ell$ be globally generated line bundles on X for which there exists $\mathbf{w} \in \mathbb{N}^\ell$ such that $\mathbf{B^w}$ is ample. If $\mathbf{B^m}$ is regular with respect to $B_1,\ldots,B_\ell$ for $\mathbf{m} \in \mathbb{N}^\ell$ and $\mathbf{B^{2m}}$ is very ample, then the line bundle $\mathbf{B^{2m+u}}$ is determinantally presented for any $\mathbf{u} \in \mathbb{N}^\ell$.*

Proof. Factor $L := \mathbf{B^{2m+u}}$ as $L = E \otimes E'$ where $E := \mathbf{B^m}$ and $E' := \mathbf{B^{m+u}}$. Theorem 2.1 in [HSS] shows that L, E, E^2, and E' are all regular with respect to $B_1,\ldots,B_\ell$. Hence, Proposition 2.5 together with Theorem 2.1 in [HSS] imply that L, E, and E' satisfy N_1, that E' has a linear free presentation with respect to E, and that both E and E^2 have a linear free presentation with respect to E'. Therefore, Theorem 3.2 proves that L is determinantally presented. $\qquad\square$

Theorem 3.2, combined with results from §2, also yields a proof for our main theorem.

Proof of Theorem 1.1. Let X be a connected scheme of dimension n and let B be a globally generated ample line bundle on X. Choose a positive integer $m \in \mathbb{N}$ such that B^m is regular with respect to B. Lemma 2.2 implies that there exists a line bundle E, which we may assume is very ample, such that, for any nef line bundle C, $E \otimes C$ is regular with respect to B. By replacing E with $E \otimes B$ if necessary, we may assume that the map $\Gamma(B) \otimes H^0(X, E \otimes B^{-1}) \longrightarrow H^0(X, E)$ is surjective. Since a sufficiently ample line bundle on X satisfies N_1 (combine Lemmata 1.1–1.3 in [I] with Fujita's Vanishing Theorem), we may also assume that, for any nef line bundle C, $E \otimes C$ satisfies N_1.

Consider the line bundle $A := E \otimes B^{2(m-1)n+1}$. If L is a line bundle on X such that $L \otimes A^{-1}$ is nef, then $L = A \otimes C = (E \otimes C) \otimes B^{2(m-1)n+1}$ for some nef line bundle C. Our choice of E guarantees that, for all $j \geqslant 1$, $(E \otimes C)^j$ is regular with respect to B, and that L satisfies N_1. Hence, Proposition 2.4 implies that $B^{2(m-1)n+1}$ has a linear free presentation with respect to $E \otimes C$. Proposition 2.5 together with Mumford's Lemma (e.g. Theorem 1.8.5 in [La2]) imply that both $E \otimes C$ and $(E \otimes C)^2$ have a linear free presentation with respect to $B^{2(m-1)n+1}$. Via Lemma 2.3 and Proposition 2.5, $B^{2(m-1)n+1}$ satisfies N_1. Therefore, Theorem 3.2 proves that L is determinantally presented. $\qquad\square$

4. Effective Bounds

In this section, we give effective bounds for determinantally presented line bundles. As a basic philosophy, one can convert explicit conditions for line bundles to satisfy N_2 into effective descriptions for determinantally presented line bundles. The three subsections demonstrate this philosophy for products of projective spaces, projective Gorenstein toric varieties, and smooth varieties. Despite not being developed here, we expect similar results for general surfaces and abelian varieties following [GP] and [Rub, PP] respectively.

4.1. Products of Projective Space. The tools from §3 lead to a description of the determinantally presented ample line bundles on a product of projective spaces. In contrast with Theorem 3.11 in [Ber] which proves that Segre-Veronese varieties are defined by

2-minors of an appropriate hypermatrix, our classification shows that a Segre-Veronese variety is typically generated by the 2-minors of a single matrix. In particular, we recover the Segre-Veronese ideals considered in [Sul, §6.2].

To study the product of projective spaces $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_\ell}$, we first introduce some notation. Let $R := \mathbb{k}[x_{i,j} : 1 \leqslant i \leqslant \ell, 0 \leqslant j \leqslant n_i]$ be the total coordinate ring (a.k.a. Cox ring) of X; this polynomial ring has the $\mathbb{Z}^\ell$-grading induced by $\deg(x_{i,j}) := e_i \in \mathbb{Z}^\ell$. Hence, we have $R_d = \Gamma(\mathscr{O}_X(d))$ for all $d \in \mathbb{Z}^\ell$ and a torus-invariant global section of $\mathscr{O}_X(d)$ is identified with a monomial $x^w \in R_d$ where $w \in \mathbb{N}^r$ and $r := \sum_{i=1}^\ell (n_i + 1)$. We write $e_{i,j}$ for the standard basis of $\mathbb{Z}^r$; in particular $x^{e_{i,j}} = x_{i,j}$.

Theorem 4.1. *Let $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_\ell}$. An ample line bundle $\mathscr{O}_X(m)$ is determinantally presented if at least $\ell - 2$ of the entries in the vector m are at least 2.*

When $\ell = 2$, this theorem shows that all of the Segre-Veronese embeddings are determinantally presented. We note that Corollary 3.3 establishes that $\mathscr{O}_X(m)$ is determinantally presented when $m_j \geqslant 2$ for all $1 \leqslant j \leqslant \ell$.

Proof. Since a line bundle $\mathscr{O}_X(v)$ is ample (and very ample) if and only if $v_j \geqslant 1$ for all $1 \leqslant j \leqslant \ell$, Corollary 1.5 in [HSS] shows that $\mathscr{O}_X(m)$ satisfies N_1. Without loss of generality, we may assume that $m_j \geqslant 2$ for $1 \leqslant j \leqslant \ell - 2$. Factor $\mathscr{O}_X(m)$ as $\mathscr{O}_X(m) = E \otimes E'$ where $u := e_1 + e_2 + \cdots + e_{\ell-1} = (1, 1, \ldots, 1, 0)$, $E := \mathscr{O}_X(u)$, and $E' := \mathscr{O}_X(m - u)$. The canonical surjection $\Gamma(E) \otimes \Gamma(E') \longrightarrow \Gamma(\mathscr{O}_X(m))$ implies that the map φ_2 in (✲) is surjective. By Lemma 3.1, it suffices prove that the map $\psi \colon Q_2 \twoheadrightarrow Q_1$ is surjective. A slight modification to the proof of Lemma 4.1 in [Stu] shows that $Q_1 = \mathrm{Ker}(\varphi_1)$ is generated by 'binomial' elements in $\mathrm{Sym}_2(\Gamma(E)) \otimes \mathrm{Sym}_2(\Gamma(E'))$ of the form $x^a x^b \otimes x^c x^d - x^{a'} x^{b'} \otimes x^{c'} x^{d'}$ where $x^a, x^b, x^{a'}, x^{b'} \in \Gamma(E)$, $x^c, x^d, x^{c'}, x^{d'} \in \Gamma(E')$, and $a + b + c + d = a' + b' + c' + d'$. Thus, the two terms in each such binomial differ by exchanging variables among the various factors. Since every such binomial element is the sum of binomials that each exchange a single pair of variables, it suffices to consider the following two cases.

In the first case, the pair of variables are exchanged between a section of E and a section E'. In particular, there exists some $1 \leqslant k \leqslant \ell - 1$ such that the binomial element has the form $x^a x^b \otimes x^c x^d - x^{a-e_{k,\alpha}+e_{k,\gamma}} x^b \otimes x^{c+e_{k,\alpha}-e_{k,\gamma}} x^d$ where $a - e_{k,\alpha}$ and $c - e_{k,\gamma}$ are nonnegative. This element is the image of $(x^a \otimes x^c)(x^b \otimes x^d) - (x^{a-e_{k,\alpha}+e_{k,\gamma}} \otimes x^{c+e_{k,\alpha}-e_{k,\gamma}})(x^b \otimes x^d)$ which lies in $Q_2 = \mathrm{Ker}(\varphi_2) \subset \mathrm{Sym}_2(\Gamma(E) \otimes \Gamma(E'))$.

In the second case, we may assume that the pair of variables are exchanged between two sections of E', as exchanging variables between two sections of E is analogous. More precisely, let $x_{k,\gamma}$ and $x_{k,\delta}$ for some $1 \leqslant k \leqslant \ell$ denote the exchanged variables and consider the binomial element $x^a x^b \otimes x^c x^d - x^a x^b \otimes x^{c-e_{k,\gamma}+e_{k,\delta}} x^{d+e_{k,\gamma}-e_{k,\delta}}$ where $c - e_{k,\gamma}$ and $d - e_{k,\delta}$ are nonnegative. Since $x^a x^b \otimes x^c x^d = x^a x^b \otimes x^d x^c$ in $\mathrm{Sym}_2(\Gamma(E)) \otimes \mathrm{Sym}_2(\Gamma(E'))$, we may also assume that $k < \ell$. Hence, there is a variable $x_{k,\alpha}$ such that $a - e_{k,\alpha}$ is nonnegative

and

$$x^a x^b \otimes x^c x^d - x^a x^b \otimes x^{c-e_{k,\gamma}+e_{k,\delta}} x^{d+e_{k,\gamma}-e_{k,\delta}}$$

$$= x^a x^b \otimes x^c x^d - x^{a-e_{k,\alpha}+e_{k,\delta}} x^b \otimes x^c x^{d+e_{k,\alpha}-e_{k,\delta}}$$

$$+ x^{a-e_{k,\alpha}+e_{k,\delta}} x^b \otimes x^c x^{d+e_{k,\alpha}-e_{k,\delta}} - x^{a-e_{k,\alpha}+e_{k,\gamma}} x^b \otimes x^{c-e_{k,\gamma}+e_{k,\delta}} x^{d+e_{k,\alpha}-e_{k,\delta}}$$

$$+ x^{a-e_{k,\alpha}+e_{k,\gamma}} x^b \otimes x^{c-e_{k,\gamma}+e_{k,\delta}} x^{d+e_{k,\alpha}-e_{k,\delta}} - x^a x^b \otimes x^{c-e_{k,\gamma}+e_{k,\delta}} x^{d+e_{k,\gamma}-e_{k,\delta}} .$$

In other words, the binomial element under consideration is a sum of binomials in which variables are exchanged between sections of E and E'. Hence, the first case shows that this binomial element lies in the image of Q_2.

We conclude that ψ is surjective and $\mathscr{O}_X(m)$ is determinantally presented. $\qquad\square$

The next proposition shows that Theorem 4.1 is optimal when $\ell = 3$. In fact, our experiments in *Macaulay2* [M2] suggest that Theorem 4.1 is always sharp.

Proposition 4.2. *If $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_\ell}$ with $\ell \geqslant 3$, then the ample line bundle $\mathscr{O}_X(1)$ is not determinantally presented.*

Proof. Any nontrivial factorization of $\mathscr{O}_X(1)$ has the form $E \otimes E'$ where $E := \mathscr{O}_X(u)$ for some $u \in \{0,1\}^\ell$ and $E' := \mathscr{O}_X(1-u)$. For a suitable choice of bases for $\Gamma(\mathscr{O}_X(u))$ and $\Gamma(\mathscr{O}_X(1-u))$, the associated matrix $\Omega(\mathscr{O}_X(u), \mathscr{O}_X(1-u))$ is the generic $(s \times t)$-matrix with $s := \sum_{u_i \neq 0}(n_i + 1)$ and $t := \sum_{i=0}^\ell (n_i + 1) - s$. Since the 2-minors of a generic $(s \times t)$-matrix define $\mathbb{P}^{s-1} \times \mathbb{P}^{t-1}$ in its Segre embedding, we see that $\mathscr{O}_X(1)$ is not determinantally presented when $\ell \geqslant 3$. $\qquad\square$

Example 4.3. Consider the variety $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ embedded in $\mathbb{P}^{11} = \mathrm{Proj}(\Bbbk[y_0, \dots, y_{11}])$ by the complete linear series of $\mathscr{O}_X(2,1,1)$. If $R = \Bbbk[x_{1,0}, x_{1,1}, x_{2,0}, x_{2,1}, x_{3,0}, x_{3,1}]$ is the total coordinate ring of X, then the twelve monomials

$$\left\{ \begin{array}{cccc} x_{1,0}^2 x_{2,0} x_{3,0}, & x_{1,0}^2 x_{2,0} x_{3,1}, & x_{1,0}^2 x_{2,1} x_{3,0}, & x_{1,0}^2 x_{2,1} x_{3,1}, \\ x_{1,0} x_{1,1} x_{2,0} x_{3,0}, & x_{1,0} x_{1,1} x_{2,0} x_{3,1}, & x_{1,0} x_{1,1} x_{2,1} x_{3,0}, & x_{1,0} x_{1,1} x_{2,1} x_{3,1}, \\ x_{1,1}^2 x_{2,0} x_{3,0}, & x_{1,1}^2 x_{2,0} x_{3,1}, & x_{1,1}^2 x_{2,1} x_{3,0}, & x_{1,1}^2 x_{2,1} x_{3,1} \end{array} \right\}$$

give an ordered basis for $\Gamma(\mathscr{O}_X(2,1,1))$. The homogeneous ideal $I_{X|\mathbb{P}^{11}}$ is the toric ideal associated to these monomials and is minimally generated by thirty three quadrics. Choosing $\{x_{1,0} x_{2,0}, x_{1,0} x_{2,1}, x_{1,1} x_{2,0}, x_{1,1} x_{2,1}\}$ and $\{x_{1,0} x_{3,0}, x_{1,0} x_{3,1}, x_{1,1} x_{3,0}, x_{1,1} x_{3,1}\}$ as ordered bases for $\Gamma(\mathscr{O}_X(1,1,0))$ and $\Gamma(\mathscr{O}_X(1,0,1))$, $\Omega(\mathscr{O}_X(1,1,0), \mathscr{O}_X(1,0,1))$ is

$$\begin{bmatrix} y_0 & y_1 & y_4 & y_5 \\ y_2 & y_3 & y_6 & y_7 \\ y_4 & y_5 & y_8 & y_9 \\ y_6 & y_7 & y_{10} & y_{11} \end{bmatrix}$$

and one may verify that the 2-minors of this matrix generates the ideal of X, so $\mathscr{O}_X(2,1,1)$ is determinantally presented. $\qquad\diamond$

However, if we consider multiple factorizations of a very ample line bundle on a product of projective spaces, then we do obtain a convenient expression of the homogeneous ideal as the 2-minors of matrices. This perspective give a conceptual explanation for both Theorem 2.6 in [Hà] and Theorem 3.11 in [Ber].

Proposition 4.4. *If $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_\ell}$, then the homogeneous ideal of X in $\mathbb{P}\big(\mathscr{O}_X(d)\big)$ is generated by the 2-minors of the matrices $\Omega\big(\mathscr{O}_X(e_i), \mathscr{O}_X(d - e_i)\big)$ where $1 \leqslant i \leqslant \ell$.*

Proof. Given Theorem 4.1, we may assume that $\ell \geqslant 3$. For brevity, set $E_i := \mathscr{O}_X(e_i)$ and $E'_i := \mathscr{O}_X(d - e_i)$ where $1 \leqslant i \leqslant \ell$. Since $\Gamma(E_i) \otimes \Gamma(E'_i)$ surjects onto $\Gamma\big(\mathscr{O}_X(d)\big)$, the map φ_2 in (✠) is surjective, and it suffices prove that the map $\psi \colon Q_2 \to Q_1$ is surjective. By an abuse of notation, we use ε_i to denote the canonical inclusion map onto the i-th summand for all three of the direct sums appearing in the middle column of (✠). As in the proof of Theorem 4.1, Q_1 is generated by binomial elements in $\bigoplus_{k=1}^{\ell} \mathrm{Sym}_2\big(\Gamma(E_k)\big) \otimes \mathrm{Sym}_2\big(\Gamma(E'_k)\big)$. Generators have the form $\varepsilon_i(x_{i,\alpha} x_{i,\beta} \otimes x^c x^d) - \varepsilon_j(x_{j,\gamma} x_{j,\delta} \otimes x^a x^b)$ where $x_{i,\alpha}, x_{i,\beta} \in \Gamma(E_i)$, $x^c, x^d \in \Gamma(E'_i)$, $x_{j,\gamma}, x_{j,\delta} \in \Gamma(E_j)$, $x^a, x^b \in \Gamma(E'_j)$ and $e_{i,\alpha} + e_{i,\beta} + c + d = a + b + e_{j,\gamma} + e_{j,\delta}$. We consider the following two cases.

In the first case, we have $i = j$. Since every binomial element is the sum of binomials that each exchange a single pair of variables, it suffices to consider an element of the form $\varepsilon_i\big(x_{i,\alpha} x_{i,\beta} \otimes x^c x^d - x_{i,\alpha} x_{i,\beta} \otimes x^{c - e_{k,\gamma} + e_{k,\delta}} x^{d + e_{k,\gamma} - e_{k,\delta}}\big)$ where $1 \leqslant k \leqslant \ell$ and both $c - e_{k,\gamma}$ and $d - e_{k,\delta}$ are nonnegative. This element is the image of

$$\varepsilon_i\big((x_{i,\alpha} \otimes x^c)(x_{i,\beta} \otimes x^d) - (x_{i,\alpha} \otimes x^{c - e_{k,\gamma} + e_{k,\delta}})(x_{i,\beta} \otimes x^{d + e_{k,\gamma} - e_{k,\delta}})\big)$$
$$- \varepsilon_k\big((x_{k,\gamma} \otimes x^{c + e_{i,\alpha} - e_{k,\gamma}})(x_{k,\delta} \otimes x^{d + e_{i,\beta} - e_{k,\delta}}) - (x_{k,\delta} \otimes x^{c + e_{i,\alpha} - e_{k,\gamma}})(x_{k,\gamma} \otimes x^{d + e_{i,\beta} - e_{k,\delta}})\big),$$

which lies in $Q_2 = \mathrm{Ker}(\varphi_2)$.

For the second case, we have $i \neq j$. We may assume that the binomial element has the form $\varepsilon_i(x_{i,\alpha} x_{i,\beta} \otimes x^c x^d) - \varepsilon_j(x_{j,\gamma} x_{j,\delta} \otimes x^{c + e_{i,\alpha} - e_{j,\gamma}} x^{d + e_{i,\beta} - e_{j,\delta}})$, where $c - e_{j,\gamma}$ and $d - e_{j,\delta}$ are nonnegative, because any additional exchanges of variables can be obtained by adding elements from the first case. This element is the image of

$$\varepsilon_i\big((x_{i,\alpha} \otimes x^c)(x_{i,\beta} \otimes x^d)\big) - \varepsilon_j\big((x_{j,\gamma} \otimes x^{c + e_{i,\alpha} - e_{j,\gamma}})(x_{j,\delta} \otimes x^{d + e_{i,\beta} - e_{j,\delta}})\big),$$

which lies in $Q_2 = \mathrm{Ker}(\varphi_2)$. $\qquad\square$

Example 4.5. Consider the variety $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ embedded in $\mathbb{P}^7 = \mathrm{Proj}(\Bbbk[y_0, \ldots, y_7])$ by the complete linear series of the line bundle $\mathscr{O}_X(1,1,1)$. The homogeneous ideal $I_{X|\mathbb{P}^7}$ is the toric ideal associated to the monomial list

$$\left\{\begin{matrix} x_{1,0} x_{2,0} x_{3,0}, & x_{1,0} x_{2,0} x_{3,1}, & x_{1,0} x_{2,1} x_{3,0}, & x_{1,0} x_{2,1} x_{3,1}, \\ x_{1,1} x_{2,0} x_{3,0}, & x_{1,1} x_{2,0} x_{3,1}, & x_{1,1} x_{2,1} x_{3,0}, & x_{1,1} x_{2,1} x_{3,1} \end{matrix}\right\}$$

and is minimally generated by nine quadrics. Choosing appropriate monomials for the ordered bases of the global sections, we obtain

$$\Omega\big(\mathscr{O}_X(1,0,0),\mathscr{O}_X(0,1,1)\big) = \begin{bmatrix} y_0 & y_1 & y_2 & y_3 \\ y_4 & y_5 & y_6 & y_7 \end{bmatrix},$$

$$\Omega\big(\mathscr{O}_X(0,1,0),\mathscr{O}_X(1,0,1)\big) = \begin{bmatrix} y_0 & y_1 & y_4 & y_5 \\ y_2 & y_3 & y_6 & y_7 \end{bmatrix},$$

$$\Omega\big(\mathscr{O}_X(0,0,1),\mathscr{O}_X(1,1,0)\big) = \begin{bmatrix} y_0 & y_2 & y_4 & y_6 \\ y_1 & y_3 & y_5 & y_7 \end{bmatrix}.$$

It follows that $\mathscr{O}_X(1,1,1)$ is not determinantally presented, but one easily verifies that the ideal $I_{X|\mathbb{P}^7}$ is generated by the 2-minors of all three matrices. $\diamond$

Multiple factorizations of a very ample line bundle allow one to describe a larger number of homogeneous ideals via 2-minors. With this in mind, it would be interesting to write down the analogue of Theorem 3.2 for multiple factorizations of the line bundle.

4.2. TORIC VARIETIES. In addition to the bound given in Corollary 3.3, there is an effective bound for toric varieties involving adjoint bundles for toric varieties; cf. Corollary 1.6 in [HSS]. Recall that a line bundle on a toric variety X is nef if and only if it is globally generated, and the dualizing sheaf K_X is a line bundle if and only if X is Gorenstein.

Proposition 4.6. *Let X be a projective n-dimensional Gorenstein toric variety with dualizing sheaf K_X, and let $B_1,\dots,B_\ell$ be the minimal generators of its nef cone $\mathrm{Nef}(X)$. Suppose that $m,m' \in \mathbb{N}^\ell$ satisfy $B^{m-u}, B^{m'-u} \in \mathfrak{B}$ for all $u \in \mathbb{N}^\ell$ with $|u| \leqslant n+1$. If $X \neq \mathbb{P}^n$ and $w \in \mathbb{N}^\ell$, then $L = K_X^2 \otimes B^{m+m'+w}$ is determinantally presented.*

Proof. Factor L as $L = E \otimes E'$ where $E := K_X \otimes B^{m+w}$ and $E' := K_X \otimes B^{m'}$. Since $B^{m-(n+1)e_j}, B^{m'-(n+1)e_j} \in \mathfrak{B}$, Corollary 0.2 in [Fu1] implies that $E \otimes B^{-e_j}$ and $E' \otimes B^{-e_j}$ belong to $\mathfrak{B}$ for all $1 \leqslant j \leqslant \ell$. For any torus invariant curve Y, there is a B^{e_j} such that $B^{e_j} \cdot Y > 0$. Theorem 3.4 in [Mus] implies that E, E^2 and E' are regular with respect to $B_1,\dots,B_\ell$. Hence, Proposition 2.5 shows that L, E, and E' satisfy N_1, and that E has a linear free presentation with respect to E', E' has a linear free presentation with respect to E, and E^2 has a linear free presentation with respect to E'. Therefore, Theorem 3.2 shows that L is determinantally presented. $\square$

Proof of Theorem 1.3 for toric varieties. This is a special case of Proposition 4.6. $\square$

We give an example showing that Theorem 1.3 is not sharp for all toric varieties.

Example 4.7. Consider the toric del Pezzo surface X obtained by blowing up $\mathbb{P}^2$ at the three torus-fixed points. Let $R := \mathbb{k}[x_0,\dots,x_5]$ be the total coordinate ring of X. The anticanonical bundle K_X^{-1} is very ample and corresponds to polygon

$$P := \mathrm{conv}\big\{(1,0),(1,1),(0,1),(-1,0),(-1,-1),(0,-1)\big\}.$$

It is easy to see that the polygon P is the smallest lattice polygon with its inner normal fan. The polygon $2P$ contains 19 lattice points. The corresponding monomials

$$\left\{\begin{array}{cccccc} x_0^4 x_1^4 x_2^2 x_5^2, & x_0^4 x_1^3 x_2 x_4 x_5^3, & x_0^4 x_1^2 x_4^2 x_5^4, & x_0^3 x_1^4 x_2^3 x_3 x_5, & x_0^3 x_1^3 x_2^2 x_3 x_4 x_5^2, \\ x_0^3 x_1^2 x_2 x_3 x_4^2 x_5^3, & x_0^3 x_1 x_3 x_4^3 x_5^4, & x_0^2 x_1^4 x_2^4 x_3^2, & x_0^2 x_1^3 x_2^3 x_3^2 x_4 x_5, & x_0^2 x_1^2 x_2^2 x_3^2 x_4^2 x_5^2, \\ x_0^2 x_1 x_2 x_3^2 x_4^3 x_5^3, & x_0^2 x_3^2 x_4^4 x_5^4, & x_0 x_1^3 x_2^4 x_3^3 x_4, & x_0 x_1^2 x_2^3 x_3^3 x_4^2 x_5, & x_0 x_1 x_2^2 x_3^3 x_4^3 x_5^2, \\ x_0 x_2 x_3^3 x_4^4 x_5^3, & x_1^2 x_2^4 x_3^4 x_4^2, & x_1 x_2^3 x_3^4 x_4^3 x_5, & x_2^2 x_3^4 x_4^4 x_5^2 & \end{array}\right\}$$

embed X into $\mathbb{P}^{18} = \mathrm{Proj}\big(\Bbbk[y_0,\ldots,y_{18}]\big)$. The homogeneous ideal $I_{X|\mathbb{P}^{18}}$ is the toric ideal associated to these monomials and is minimally generated by 129 quadrics. Choosing $\{x_0^2 x_1^2 x_2 x_5, x_0^2 x_1 x_4 x_5^2, x_0 x_1^2 x_2^2 x_3, x_0 x_1 x_2 x_3 x_4 x_5, x_0 x_3 x_4^2 x_5^2, x_1 x_2^2 x_3^2 x_4, x_2 x_3^2 x_4^2 x_5\}$ as an ordered basis for $\Gamma(K_X^{-1})$, the matrix $\Omega\big(K_X^{-1}, K_X^{-1}\big)$ is

$$\begin{bmatrix} y_0 & y_1 & y_3 & y_4 & y_5 & y_8 & y_9 \\ y_1 & y_2 & y_4 & y_5 & y_6 & y_9 & y_{10} \\ y_3 & y_4 & y_7 & y_8 & y_9 & y_{12} & y_{13} \\ y_4 & y_5 & y_8 & y_9 & y_{10} & y_{13} & y_{14} \\ y_5 & y_6 & y_9 & y_{10} & y_{11} & y_{14} & y_{15} \\ y_8 & y_9 & y_{12} & y_{13} & y_{14} & y_{16} & y_{17} \\ y_9 & y_{10} & y_{13} & y_{14} & y_{15} & y_{17} & y_{18} \end{bmatrix},$$

and its 2-minors generate $I_{X|\mathbb{P}^{18}}$. However, Theorem 1.3 only establishes that the line bundle $K_X^{-4} = K_X^2 \otimes (K_X^{-1})^{2\cdot 2 + 2}$ is determinantally presented. $\diamond$

4.3. SMOOTH VARIETIES. For smooth varieties, we also have an effective bound for adjoint bundles; see Theorem 1.3.

Proof of Theorem 1.3 for smooth varieties. Factor the line bundle L as $L = E \otimes E'$ where $E := K_X \otimes A^{n+1}$ and $E' := K_X \otimes A^{j-n-1} \otimes B$. Since $j \geqslant 2n+2$ and E is nef (see Example 1.5.35 in [La2]), Proposition 2.6 implies that L, E, and E' satisfy N_1, E has a linear free presentation with respect to E', E' has a linear free presentation with respect to E, and E^2 has a linear free presentation with respect to E'. Therefore, Theorem 3.2 shows that L is determinantally presented. $\square$

We end with an example showing that the hypotheses in Theorem 1.3 are optimal without further restrictions on the varieties under consideration.

Example 4.8. Consider the Grassmannian $X = \mathrm{Gr}(2,4)$ parametrizing all two dimensional subspaces of the vector space $\Bbbk^4$. Let $\mathscr{O}_X(1)$ denote the determinant of the universal rank 2 sub-bundle on X. The associated complete linear series determines the Plücker embedding of X into $\mathbb{P}^5 = \mathrm{Proj}(\Bbbk[x_{1,2}, x_{1,3}, x_{1,4}, x_{2,3}, x_{2,4}, x_{3,4}])$. Since $I_{X|\mathbb{P}^5} = \langle x_{1,2} x_{3,4} - x_{1,3} x_{2,4} + x_{2,3} x_{1,4} \rangle$, it follows that $\mathscr{O}_X(1)$ is not determinantally presented. On the other hand, the monomials

$$\left\{\begin{array}{ccccccc} x_{1,2}^2, & x_{1,2} x_{1,3}, & x_{1,2} x_{1,4}, & x_{1,2} x_{2,3}, & x_{1,2} x_{2,4}, & x_{1,2} x_{3,4}, & x_{1,3}^2, \\ x_{1,3} x_{1,4}, & x_{1,3} x_{2,3}, & & x_{1,3} x_{3,4}, & x_{1,4}^2, & x_{1,4} x_{2,3}, & x_{1,4} x_{2,4}, \\ x_{1,4} x_{3,4}, & x_{2,3}^2, & x_{2,3} x_{2,4}, & x_{2,3} x_{3,4}, & x_{2,4}^2, & x_{2,4} x_{3,4}, & x_{3,4}^2 \end{array}\right\}$$

form an ordered basis for $\Gamma\big(\mathscr{O}_X(2)\big)$, so the complete linear series of $\mathscr{O}_X(2)$ embeds X into $\mathbb{P}^{19} = \mathrm{Proj}(\Bbbk[y_0,\dots,y_{19}])$. The matrix $\Omega\big(\mathscr{O}_X(1),\mathscr{O}_X(1)\big)$ is

$$\begin{bmatrix} y_0 & y_1 & y_2 & y_3 & y_4 & y_5 \\ y_1 & y_6 & y_7 & y_8 & y_5 + y_{11} & y_9 \\ y_2 & y_7 & y_{10} & y_{11} & y_{12} & y_{13} \\ y_3 & y_8 & y_{11} & y_{14} & y_{15} & y_{16} \\ y_4 & y_5 + y_{11} & y_{12} & y_{15} & y_{17} & y_{18} \\ y_5 & y_9 & y_{13} & y_{16} & y_{18} & y_{19} \end{bmatrix}$$

and the 2-minors of this matrix generated $I_{X|\mathbb{P}^{19}}$ (indeed, this is the second Veronese of the Plücker embedding). Since $K_X = \mathscr{O}_X(-4)$ and $\mathscr{O}_X(2) = K_X^2 \otimes \mathscr{O}_X(1)^{2\cdot 4+2}$, we see that the bound in Theorem 1.3 is sharp in this case.   ⋄

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